HAMILTONIAN VECTOR FIELDS OF HOMOGENEOUS POLYNOMIALS IN TWO VARIABLES

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Abstract. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p \geq 2$, $G = (-g'_y, g'_x)$ be its Hamiltonian vector field, and $\mathbf{G}_t$ be the local flow generated by G. Denote by $\mathcal{E}(G, O)$ the space of germs of C^∞ diffeomorphisms $(\mathbb{R}^2, O) \to (\mathbb{R}^2, O)$, that preserve orbits of G. Let also $\hat{\mathcal{E}}_{\mathrm{id}}(G, O)$ be the identity component of $\hat{\mathcal{E}}(G, O)$ with respect to C^1 topology.

Suppose that g has no multiple prime factors. Then we prove that for every $h \in \hat{\mathcal{E}}_{\mathrm{id}}(G, O)$ there exists a germ of a smooth function $\alpha : \mathbb{R}^2 \to \mathbb{R}$ at O such that

$$h(z) = \mathbf{G}_{\alpha(z)}(z).$$

1. Introduction

Let $p \geq 1$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p+1$, i.e. $\deg g \geq 2$. Then we have a prime decomposition of g over $\mathbb{R}$:

$$g(x, y) = \prod_{i=1}^{l} L_i(x, y) \cdot \prod_{j=1}^{p+1-l} Q_j(x, y), \tag{1.1}$$

where every $L_i = a_i x + b_i y$ is a linear function, and every Q_j is a definite quadratic form.

Lemma 1.1. [5] *The following conditions for a homogeneous polynomial g of degree $\deg g \geq 2$ are equivalent:*

(1) *decomposition* (1.1) *contains no multiple factors*
(2) *none of the partial derivatives g'_x and g'_y is identically zero (i.e. g does depend on x and y) and these polynomials are relatively simple in the ring $\mathbb{R}[x, y]$.*

In this case the origin $O \in \mathbb{R}^2$ is a unique critical point for g.

Definition 1.2 (Property $(*)$ for a polynomial)**.** *Say that a homogeneous polynomial $g \in \mathbb{R}[x, y]$ of degree $\deg g \geq 2$ has property $(*)$ if it satisfies one of the conditions of Lemma 1.1.*

Example 1.3. *For $n \geq 2$ consider the following function*

$$\omega_n : \mathbb{C} \to \mathbb{C}, \qquad \omega_n(z) = z^n.$$

Then its real and imagine parts $\mathrm{Re}(z^n)$ *and* $\mathrm{Im}(z^n)$ *have property* $(*)$*.*

Let $H = (-g'_y, g'_x)$ be the Hamiltonian vector field for g. Then g is constant along orbits of H. The typical foliations of $\mathbb{R}^2$ by level sets of homogeneous polynomials are shown in Figures 4.1 and 4.2.

Notice that the property $(*)$ for g can be formulated as follows: *the Hamiltonian vector field H of g can not be represented as a product $H = \omega H_1$, where ω is a homogeneous polynomial of degree* $\deg \omega \geq 1$ *and H_1 is a homogeneous vector field.*

Definition 1.4 (Property $(*)$ for a vector field)**.** *Say that a vector field G on $\mathbb{R}^2$* ***has property*** $(*)$ ***at*** *O if there exist a smooth (C^∞) and everywhere non-zero function $\eta : \mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$, local coordinates (x, y) at O, and a homogeneous polynomial $g(x, y)$ having property $(*)$ such that*

$$G = \eta H,$$

where $H = (-g_y, g_x)$ is a Hamiltonian vector field of g.

It follows from Lemma 1.1 that in this case the origin $O \in \mathbb{R}^2$ is an isolated singular point of G.

1.5. **Main result.** Let G be a smooth vector field defined in a neighborhood of the origin $O \in \mathbb{R}^2$. Denote by $\hat{\mathcal{E}}(G, O)$ the set of germs of C^∞ diffeomorphisms

$$h : (\mathbb{R}^2, O) \to (\mathbb{R}^2, O)$$

preserving orbits of G, i.e. $h \in \hat{\mathcal{E}}(G, O)$ if there exists a neighborhood V of O such that

$$h(\omega \cap V) \subset \omega \tag{1.2}$$

for each orbit ω of G.

Let also $\hat{\mathcal{E}}_{\mathrm{id}}(G, O)$ be the *identity component* of $\hat{\mathcal{E}}(G, O)$ with respect to C^1-topology. It consists of germs of diffeomorphisms at O isotopic to $\mathrm{id}_{\mathbb{R}^2}$ in $\hat{\mathcal{E}}(G, O)$ via isotopy whose partial derivatives of the first order continuously depend on the parameter, see [5] for details.

Denote by $\mathbf{G} : \mathbb{R}^2 \times \mathbb{R} \supset \mathcal{U}_{\mathbf{G}} \longrightarrow \mathbb{R}^2$ the corresponding local flow of G defined on an open neighborhood $\mathcal{U}_{\mathbf{G}}$ of $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^2 \times \mathbb{R}$.

Then for every germ of a smooth function $\alpha : \mathbb{R}^2 \to \mathbb{R}$ at O we can define the following map $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h(z) = \mathbf{G}(z, \alpha(z)). \tag{1.3}$$

This map will be called the *smooth shift* along orbits of G via the function α. Denote by $Sh(G, O)$ the set of germs of mappings of the form (1.3), where α runs over all germs of smooth function at O.

Then, see [4], $Sh(G, O) \subset \hat{\mathcal{E}}_{\mathrm{id}}(G, O)$.

In this paper we prove the following theorem:

Theorem 1.6. *Let G be a vector field on $\mathbb{R}^2$ having property $(*)$ at O. Then $Sh(G, O) = \hat{\mathcal{E}}_{\mathrm{id}}(G, O)$. Thus every $h \in \hat{\mathcal{E}}_{\mathrm{id}}(G, O)$ can be represented in the form* (1.3) *for some smooth function $\alpha : \mathbb{R}^2 \to \mathbb{R}$.*

Remark 1.7. Suppose that O is a regular point for G, i.e. $G(O) \neq 0$. Then every smooth map preserving orbits of G is a neighborhood of O is a shift along orbits of G via a certain *smooth* function α. For the convenience of the reader we recall a proof of this fact, see [4, Eq. (10)]. Indeed, since $G(O) \neq 0$, it follows that there are local coordinates $(x_1, \ldots, x_n)$ at O such that $G(x) = (1, 0, \ldots, 0)$, whence

$$\mathbf{G}(x_1, \ldots, x_n, t) = (x_1 + t, x_2, \ldots, x_n).$$

If now $h = (h_1, \ldots, h_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map that preserves orbits of G, then $h_i = x_i$ for $2 \leq i \leq n$. Set

$$\alpha(x) = h_1(x) - x_1. \tag{1.4}$$

Then $h(x) = \mathbf{G}(x, \alpha(x))$.

1.8. **Applications.** In [4] the identity

$$Sh(G, O) = \hat{\mathcal{E}}_{\mathrm{id}}(G, O)$$

is established for all linear vector fields on $\mathbb{R}^n$. Thus if $G(x) = A \cdot x$ is a linear vector field on $\mathbb{R}^n$, where A is a non-zero $(n \times n)$-matrix, then every $h \in \hat{\mathcal{E}}_{\mathrm{id}}(G, O)$ can be represented as follows

$$h(x) = e^{\alpha(x)A} \cdot x$$

for a certain smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}$. It allowed for a vector field G satisfying mild conditions describe the homotopy types of the connected components of the group $\mathcal{D}(G)$ of orbit preserving diffeomorphisms. This result was essentially used in [3] for the calculation of the homotopy types of stabilizers and orbits of Morse functions on compact surfaces M with respect to the action of $\mathcal{D}(M)$.

Theorem 1.6 allowed to perform similar calculation for large class of functions on surfaces with isolated singularities. This will be done in another paper.

1.9. **Structure of the paper.** In Section 2 the definition of weak Whitney topologies is given.

Section 3 includes a plan of the proof of Theorem 1.6. Using results of [5] the proof is reduced to the case when $h \in \hat{\mathcal{E}}_{\mathrm{id}}(G, O)$ is ∞-close to he identity at O, see Proposition 3.4. It turns out that in order to work with these mappings it is convenient to use polar coordinates (ϕ, ρ), see Section 4. In this case instead of a unique singular point $O = (0,0) \in \mathbb{R}^2$ we obtain a whole line of singular points $\rho = 0$, but the formulas for the vector field G in polar coordinates becomes essentially simple.

Then in Section 5 it is shown that instead of smooth functions on $\mathbb{R}^2$ that are flat at O, we can consider smooth functions with respect to polar coordinates (ϕ, ρ) being flat for $\rho = 0$. Similarly, in Section 6 it is proved that instead of diffeomorphisms of $\mathbb{R}^2$ that are ∞-close to the identity at O it is possible to consider diffeomorphisms of the half-plane of polar coordinates $\mathbb{H}$ that are ∞-close to the identity for $\rho = 0$.

In Section 7 a proof of Proposition 3.4 is given. This will complete Theorem 1.6.

2. Continuous maps between functional spaces

Let $V \subset \mathbb{R}^n$ be an open subset and $f = (f_1, \dots, f_m) : V \to \mathbb{R}^m$ be a smooth mapping. For every compact $K \subset V$ and integer $r \geq 0$ define the r-*norm* of f on K by

$$\|f\|^r_K \;=\; \sum_{j=1}^{m} \sum_{|i| \leq r} \sup_{x \in K} |D^i f_j(x)|,$$

where $i = (i_1, \dots, i_n)$, $|i| = i_1 + \cdots + i_n$, and $D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$. For a fixed r the norms $\|f\|^r_K$ define the so-called *weak* C^r_W Whitney topology on $C^\infty(V, \mathbb{R}^m)$, see [1, 2].

Definition 2.1. *Let A, B, C, D be smooth manifolds,*

$$\mathcal{X} \subset C^\infty(A, B), \qquad \mathcal{Y} \subset C^\infty(C, D)$$

*be two subsets and $F : \mathcal{X} \to \mathcal{Y}$ be a map. Say that F is $C^{s,r}_{W,W}$-**continuous** provided it is continuous from C^s_W-topology on $\mathcal{X}$ to C^r_W-topology on $\mathcal{Y}$.*

*Say that F is **tamely continuous** if for every $r \geq 0$ there exists an integer number $s(r) \geq 0$ such that F is $C^{s(r),r}_{W,W}$-continuous. Evidently, every tamely continuous map is $C^{\infty,\infty}_{W,W}$-continuous.*

The following lemmas are easy to prove, see [5].

Lemma 2.2. *Let $D : C^\infty(V) \to C^\infty(V)$ be the mapping defined by*
$$D(\alpha) = \frac{\partial^{|k|}\alpha}{\partial x^k},$$
where $k = (k_1, \dots, k_n)$, $|k| = \sum\limits_{i=1}^{n} k_i$, and $\partial x^k = \partial x_1^{k_1} \cdots \partial x_n^{k_n}$. Then D is $C^{r+|k|,r}_{W,W}$-continuous for all $r \geq 0$.

Lemma 2.3. *Let $Z : C^\infty(V) \to C^\infty(V)$ be the mapping defined by*
$$Z(\alpha)(x_1, \dots, x_n) = x_1 \cdot \alpha(x_1, \dots, x_n), \qquad \alpha \in C^\infty(V).$$
Then Z is injective and for every $r \geq 0$ the mapping Z is $C^{r,r}_{W,W}$-continuous and its inverse Z^{-1} is a $C^{r+1,r}_{W,W}$-continuous.

Lemma 2.4 (Hadamard)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(0) = 0$, then $f(x) = s \underbrace{\int_0^1 f'(tx)dt}_{g(x)} = x\, g(x)$, where g is smooth and $g(0) = f'(0)$.* $\square$

More generally,
$$f(x + y) = f(x) + y \underbrace{\int_0^1 f'(x + sy)ds}_{g(x,y)} = f(x) + y \cdot g(x, y), \tag{2.1}$$
where g is also smooth and such that $g(x, 0) = f'(x)$.

In particular, if f has an inverse function f^{-1} then
$$f(f^{-1}(x)+y) = f(f^{-1}(x))+y\cdot g(f^{-1}(x), y) = x+y\cdot g(f^{-1}(x), y). \tag{2.2}$$

3. Proof of Theorem 1.6

Actually we establish a more general statement. First we introduce some notation.

3.1. **Smooth shifts along orbits of vector fields.** Let G be a vector field on a manifold M. We will always denote by
$$\mathbf{G} : M \times \mathbb{R} \supset \mathcal{U}_{\mathbf{G}} \to M$$
the local flow of G, where $\mathcal{U}_{\mathbf{G}}$ is an open neighborhood of $M \times 0$ in $M \times \mathbb{R}$.

For every open subset $V \subset M$ let
$$\mathcal{E}(G, V) \ \subset \ C^\infty(V, M)$$
be the set of all smooth mappings $h : V \to M$ such that

(1) $h(\omega \cap V) \subset \omega$ for every orbit ω of G, in particular h is fixed on the set of singular points of G contained in V;
(2) h is a local diffeomorphism at every singular point of G.

Let also $\mathcal{E}_{\mathrm{id}}(G,V)$ be the subset of $\mathcal{E}(G,V)$ consisting of mappings h such that

(3) h is homotopic to id_M in $\mathcal{E}(G,V)$.

If $V = M$, then $\mathcal{E}(G,M)$ and $\mathcal{E}_{\mathrm{id}}(G,M)$ will be denoted by $\mathcal{E}(G)$ and $\mathcal{E}_{\mathrm{id}}(G)$ respectively.

Let $O \in V$ be a singular point of G. Then $h(O) = O$ for every $h \in \mathcal{E}(G,V)$. Denote by $\mathcal{E}_{\infty}(G,V,O)$ the subset of $\mathcal{E}(G,V)$ consisting of mappings h which are ∞-*close to the identity at* O, i.e. the ∞-jets of h and id_V at O coincide.

Theorem 3.2. *Let G be a vector field on $\mathbb{R}^2$ having property $(*)$ at O and V be a sufficiently small open neighborhood of O. Then for every $f \in \mathcal{E}_{\mathrm{id}}(G,V)$ there exists a neighborhood $\mathcal{U}_f$ in $\mathcal{E}_{\mathrm{id}}(G)$ with respect to C^p_W-topology and a tamely continuous map*

$$\sigma_V \ : \ \mathcal{E}_{\mathrm{id}}(G,V) \supset \mathcal{U}_f \ \longrightarrow \ C^{\infty}(V)$$

such that

$$h(z) = \mathbf{G}(z, \sigma_V(h)(z))$$

for every $h \in \mathcal{U}_f$.

Moreover, if $\deg g \geq 3$, then σ can be defined on all of $\mathcal{E}_{\mathrm{id}}(G,V)$.

The proof is based on the following two statements. The first one is established in [5]:

Proposition 3.3. [5] *Let G be a vector field on $\mathbb{R}^2$ having property $(*)$ at O and $U \subset V$ be two sufficiently small open neighborhoods of O. Then for every $f \in \mathcal{E}_{\mathrm{id}}(G,V)$ there exists a neighborhood $\mathcal{U}_f$ in $\mathcal{E}_{\mathrm{id}}(G,V)$ with respect to C^p_W-topology and a tamely continuous map*

$$\Lambda : \mathcal{U}_f \to C^{\infty}(V)$$

such that for every $h \in \mathcal{U}_f$ we have that

$$\operatorname{supp} \Lambda(h) \subset U$$

and the mapping $\hat{h} : V \to \mathbb{R}$ defined by

$$\hat{h}(z) = \mathbf{G}(h(z), -\Lambda(h)(z))$$

is ∞-close to $\mathrm{id}_{\mathbb{R}^2}$ at O. In particular, $\hat{h} \in \mathcal{E}_{\infty}(G,V,O)$.

Moreover, if $\deg g \geq 3$, then Λ can be defined on all of $\mathcal{E}_{\mathrm{id}}(G)$.

The second statement will be proved in Section 7.

Proposition 3.4. *Let G be a vector field on $\mathbb{R}^2$ having property $(*)$ at O and V be a sufficiently small open neighborhood of O. Then there exists a unique map*

$$\Psi : \mathcal{E}_\infty(G, V, O) \to \mathrm{Flat}\,(\mathbb{R}^2, O)$$

such that for every $\hat{h} \in \mathcal{E}_\infty(G, V, O)$ we have that

$$\hat{h}(z) = \mathbf{G}(z, \Psi(\hat{h})(z)) \tag{3.1}$$

This mapping is $C^{3r+p,r}_{W,W}$-continuous for every $r \geq 0$.

Now we can complete Theorem 3.2. First notice that for a smooth function α and a mapping h the following relations are equivalent:

$$h(z) = \mathbf{G}(z, \alpha(z)) \qquad \text{and} \qquad z = \mathbf{G}(h(z), -\alpha(z)). \tag{3.2}$$

Let $f \in \mathcal{E}_{\mathrm{id}}(G)$. Then it follows from Proposition 3.3 that for every $f \in \mathcal{E}_{\mathrm{id}}(G)$ there exists a C^p_W-neighborhood $\mathcal{U}_f$ of f in $\mathcal{E}_{\mathrm{id}}(G)$ and a well-defined map

$$H : \mathcal{U}_f \to \mathcal{E}_\infty(G, V, O)$$

given by

$$H(h)(z) = \mathbf{G}(h(z), -\Lambda(h)(z)).$$

Then the following map $\sigma : \mathcal{U}_f \to C^\infty(V)$ defined by

$$\sigma = \Lambda + \Psi \circ H$$

satisfies the statement of our theorem.

Indeed, let $h \in \mathcal{U}_f$ and $\hat{h} = H(h)$. Then

$$\sigma(h) = \Lambda(h) + \Psi \circ H(h) = \Lambda(h) + \Psi(\hat{h}).$$

Whence

$$\begin{aligned} \mathbf{G}\big(h(z), -\sigma(h)(z)\big) &= \mathbf{G}\big(h(z), -\Lambda(h)(z) - \Psi(\hat{h})(z)\big) = \\ &= \mathbf{G}\Big(\underbrace{\mathbf{G}\big(h(z), -\Lambda(h)(z)\big)}_{\hat{h}}, -\Psi(\hat{h})(z) \Big) = \\ &= \mathbf{G}\big(\hat{h}(z), -\Psi(\hat{h})(z)\big) \xlongequal{(3.1)\quad(3.2)} z, \end{aligned}$$

Therefore

$$h(z) = \mathbf{G}(z, \sigma(h)(z)).$$

If $\deg g \geq 3$, then σ is defined on all of $\mathcal{E}_{\mathrm{id}}(G)$.

Theorem 3.2 is completed modulo Proposition 3.4. The proof of this proposition will be given in Section 7.

4. Polar coordinates

Let $\mathbb{H} = \{(\phi, \rho) \mid \rho \geq 0\} \subset \mathbb{R}^2$ be the closed upper half-plane of $\mathbb{R}^2$ with cartesian coordinates which we denote by (ϕ, ρ). Let also $\partial\mathbb{H} = \{\rho = 0\}$ be its boundary (i.e. ϕ-axis), and $\overset{\circ}{\mathbb{H}} = \{\rho > 0\}$ the interior of $\mathbb{H}$. Take another copy of $\mathbb{R}^2$ with coordinates (x, y) and consider the following map

$$P_k : \mathbb{H} \to \mathbb{R}^2, \qquad P_k(\phi, \rho) = (\rho^k \cos\phi, \rho^k \sin\phi).$$

For $k = 1$ this map defines the so-called *polar* coordinates in $\mathbb{R}^2$. We will also denote the mapping P_1 simply by P.

Evidently, $P_k(\partial\mathbb{H}) = 0 \in \mathbb{R}^2$ and the restriction of P_k onto $\overset{\circ}{\mathbb{H}}$ is a $\mathbb{Z}$-covering map: $P_k : \overset{\circ}{\mathbb{H}} \to \mathbb{R}^2 \setminus \{O\}$, where the group $\mathbb{Z}$ acts on $\mathbb{H}$ by $n \cdot (\phi, \rho) = (\phi + 2\pi n, \rho)$.

Lemma 4.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p + 1$ and $\phi_0 \in \mathbb{R}$. Then there are unique but not necessarily distinct numbers ϕ_i, $(i = 1, \dots, l)$ such that*

$$\phi_0 - \frac{\pi}{2} \ \leq\ \phi_1 \ \leq\ \dots\ \leq \phi_l \ <\ \phi_0 + \frac{\pi}{2}$$

and a smooth function γ such that $\gamma(\phi) \neq 0$ for all $\phi \in (\phi_0 - \frac{\pi}{2}, \phi_0 + \frac{\pi}{2})$ and

$$g(\rho\cos\phi, \rho\sin\phi) = \rho^{p+1} \cdot \gamma(\phi) \cdot \prod_{i=1}^{l} (\phi - \phi_i).$$

Proof. Notice that there exists a unique decomposition of g:

$$g(x, y) = \tau(x, y) \cdot \prod_{i=1}^{l} (b_i x + a_i y), \tag{4.1}$$

where

$$a_i = \cos\phi_i, \qquad b_i = \sin\phi_i,$$

for a unique $\phi_i \in [\phi_0 - \frac{\pi}{2}, \phi_0 + \frac{\pi}{2})$, $(i = 1, \dots, l)$, such that $\phi_i \leq \phi_{i+1}$, and τ is a homogeneous polynomial of degree $p + 1 - l$ such that

$$\tau(x, y) \neq 0, \qquad \text{for } (x, y) \neq 0.$$

Therefore

$$b_i x + a_i y = \sin\phi_i \cdot \rho\cos\phi + \cos\phi_i \cdot \rho\sin\phi = \rho \cdot \sin(\phi - \phi_i),$$

and thus

$$g(\rho\cos\phi, \rho\sin\phi) = \rho^{p+1} \cdot \tau(\cos\phi, \sin\phi) \cdot \prod_{i=1}^{l} \sin(\phi - \phi_i).$$

Notice that the function $\frac{\sin(\phi-\phi_i)}{(\phi-\phi_i)}$ is smooth and strictly positive on the interval $(\phi_i - \pi, \phi_i + \pi)$ and $\tau(\cos\phi, \sin\phi) > 0$ for every ϕ, we obtain that

$$g(\rho\cos\phi, \rho\sin\phi) = \rho^{p+1} \cdot \gamma(\phi) \cdot \prod_{i=1}^{l} (\phi - \phi_i),$$

for a certain smooth function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\gamma(\phi) \neq 0$ for all $\phi \in (\phi_0 - \frac{\pi}{2}, \phi_0 + \frac{\pi}{2})$. $\square$

The level curves of a homogeneous polynomial $g : \mathbb{R}^2 \to \mathbb{R}$ and the mapping $g \circ P_k : \mathbb{H} \to \mathbb{R}$ are shown in Figure 4.1 for $l = 0$ and in Figure 4.2 for $l \geq 1$.

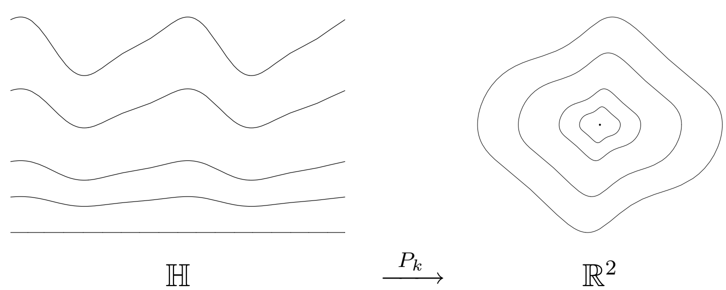

FIGURE 4.1. $l = 0$.

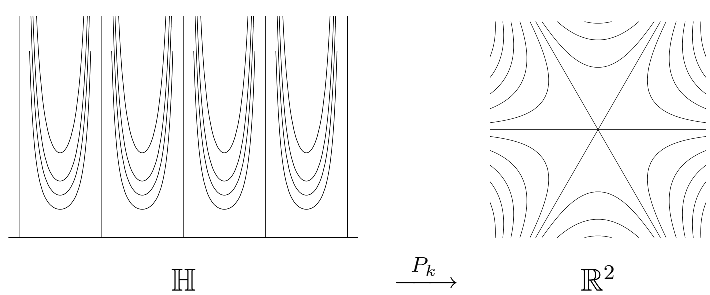

FIGURE 4.2. $l \geq 1$.

4.2. **Lifting vector fields from $\mathbb{R}^2$ to $\mathbb{H}$.** Let G be a smooth vector field defined in a neighborhood V of $O \in \mathbb{R}^2$. Denote

$$U = P_k^{-1}(V) \subset \mathbb{H}.$$

If $G(O) = 0$, then there exists a unique $\mathbb{Z}$-invariant vector field F on U vanishing on $\partial\mathbb{H}$, and such that the following diagram is commutative:

$$
\begin{array}{ccccccc}
 & & TU & \xrightarrow{TP_k} & TV & & \\
 & & {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle G} & & \\
\mathbb{H} & \supset & U & \xrightarrow{P_k} & V & \subset & \mathbb{R}^2
\end{array}
\tag{4.2}
$$

Notice that in general F is smooth only on $U \cap \overset{\circ}{\mathbb{H}}$ and is just *continuous* on $\mathbb{H}$.

Let $\mathbf{F}_t$ and $\mathbf{G}_t$ be the local flows generated by F and G respectively. Then for every $t \in \mathbb{R}$ the following diagram is commutative

$$
\begin{array}{ccc}
U & \xrightarrow{\mathbf{F}_t} & \mathbb{H} \\
{\scriptstyle P_k}\downarrow & & \downarrow{\scriptstyle P_k} \\
V & \xrightarrow{\mathbf{G}_t} & \mathbb{R}^2
\end{array}
\qquad \text{i.e.} \qquad P_k \circ \mathbf{F}_t(x) = \mathbf{G}_t \circ P_k(x),
\tag{4.3}
$$

provided both parts of this equality are defined.

The following lemma is crucial for the proof of Proposition 3.4.

Lemma 4.3. *If $a, a' \in U$ belong to the same orbit of $\mathbf{F}$, then $b = P_k(a)$ and $b' = P_k(a')$ belong to the same orbit of $\mathbf{G}$, see Figure 4.3. Moreover, the time between a and a' with respect to $\mathbf{F}$ is equal to the time between b and b' with respect to $\mathbf{G}$.*

Proof. Indeed, if $a' = \mathbf{F}_\tau(a)$, then

$$b' = P_k(a') = P_k \circ \mathbf{F}_\tau(a) = \mathbf{G}_\tau \circ P_k(a) = \mathbf{G}_\tau(b).$$

Lemma is proved. □

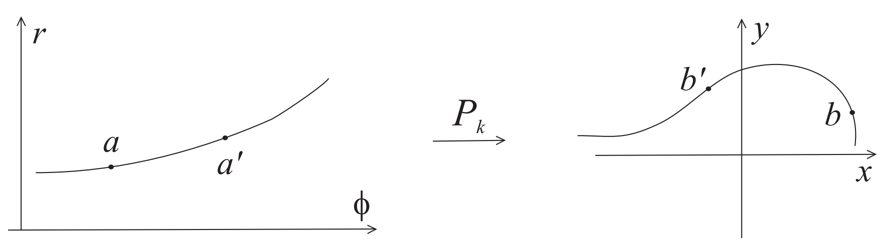

FIGURE 4.3.

Lemma 4.4. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p + 1 \geq 2$, $H = (-g'_y, g'_x)$ be the Hamiltonian vector field of g, and*

$$\eta : \mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$$

a smooth everywhere non-zero function. Consider the following vector field

$$G = \eta H = \eta \cdot (-g'_y, g'_x)$$

and let $F = (F_1, F_2)$ be the vector field on $\mathbb{H}$ induced by G via mapping

$$P_1 = P : \mathbb{H} \to \mathbb{R}^2, \qquad P(\phi, \rho) = (\rho\cos\phi, \rho\sin\phi).$$

Write g in the following form

$$g(x, y) = y^a R(x, y),$$

where $a \geq 0$ and R is a polynomial that is not divided by y. Then

$$F_1(\phi, \rho) = \frac{(p+1) \cdot g(P(\phi, \rho))}{\rho^2} = \rho^{p-1}\, \phi^a\, \gamma_1(\phi), \tag{4.4}$$

for a certain smooth function $\gamma_1 : \mathbb{R} \to \mathbb{R}$ such that $\gamma_1(0) \neq 0$.

Moreover, if $a \geq 1$, then

$$F_2(\phi, \rho) = \rho^p\, \phi^{a-1}\, \gamma_2(\phi), \tag{4.5}$$

where $\gamma_2 : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\gamma_2(0) \neq 0$.

Corollary 4.5. *If g has property $(*)$, then $a = 0$ or 1. Hence*

$$F_1(\phi, \rho) = \rho^{p-1}\gamma_1(\phi), \quad \text{if } a = 0,$$
$$F_2(\phi, \rho) = \rho^p\gamma_2(\phi), \quad \text{if } a = 1.$$

Thus in both cases one of the coordinate functions of F does not divides by ϕ.

Proof of Lemma 4.4. First notice that for a homogeneous polynomial g of degree $p+1$ the following *Euler identity* holds true:

$$xg'_x + yg'_y = (p+1)g. \tag{4.6}$$

Also, it follows from Lemma 4.1 that every multiple y in g yields the multiple ϕ in $g \circ P$. Therefore

$$g \circ P(\phi, \rho) = \rho^{p+1}\, \phi^a\, \gamma_1(\phi), \tag{4.7}$$

for a certain smooth function $\gamma_1 : \mathbb{R} \to \mathbb{R}$ such that $\gamma_1(0) \neq 0$.

Consider now the Jacobi matrix of P:

$$J(P) = \begin{pmatrix} -\rho\sin\phi & \cos\phi \\ \rho\cos\phi & \sin\phi \end{pmatrix}$$

Then it follows from the commutative diagram (4.2) that

$$G \circ P = J(P) \cdot F,$$

i.e.

$$\begin{pmatrix} G_1 \circ P \\ G_2 \circ P \end{pmatrix} = \begin{pmatrix} -\rho\sin\phi & \cos\phi \\ \rho\cos\phi & \sin\phi \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

whence

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\rho}\sin\phi & \frac{1}{\rho}\cos\phi \\ \cos\phi & \sin\phi \end{pmatrix} \cdot \begin{pmatrix} G_1 \circ P \\ G_2 \circ P \end{pmatrix}.$$

Therefore

$$F_1 = \frac{-(G_1 \circ P) \cdot \sin\phi + (G_2 \circ P) \cdot \cos\phi}{\rho}.$$

Denote

$$A(x,y) = \frac{-yG_1 + xG_2}{x^2+y^2} = \frac{yg'_y + xg'_x}{x^2+y^2} \cdot \eta \overset{(4.6)}{=\!=\!=} \frac{(p+1)g}{x^2+y^2} \cdot \eta.$$

Then

$$F_1 = A \circ P \overset{(4.7)}{=\!=\!=} \rho^{p-1}\, \phi^a\, \gamma_1(\phi).$$

Similarly,

$$F_2 = (G_1 \circ P) \cdot \cos\phi + (G_2 \circ P) \cdot \sin\phi.$$

Put

$$B(x,y) = \frac{xG_1 + yG_2}{\sqrt{x^2+y^2}} = \frac{-xg'_y + yg'_x}{\sqrt{x^2+y^2}} \cdot \eta.$$

Then $F_2 = B \circ P$. Since the numerator of the latter fraction is a homogeneous polynomial of degree $p+1$, it follows from Lemma 4.1 that

$$F_2 = \rho^p\, \phi^{a_1}\, \gamma_2(\phi),$$

for certain $a_1 \geq 0$ and a smooth function $\gamma_2 : \mathbb{R} \to \mathbb{R}$ such that $\gamma_2(0) \neq 0$.

It remains to prove that if $a \geq 1$ then

$$a_1 = a - 1.$$

Equivalently, we have to show that the numerator:

$$N = -xg'_y + yg'_x$$

of B is divided by y^{a-1} but not by y^a.

Notice that

$$g'_x = y^a R'_x, \qquad g'_y = ay^{a-1}R + y^a R'_y.$$

Whence

$$N = -xg'_y + yg'_x = -axy^{a-1}R - xy^a R'_y + y^{a+1}R'_x$$

Since R is not divided by y, it follows that N is divided by y^{a-1} but not by y^a. $\square$

5. Correspondence between flat functions

Recall that a smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}$ is *flat* on a subset $K \subset \mathbb{R}^n$ provided all partial derivatives of α of all orders vanish at avery point $x \in K$.

Let $\mathrm{Flat}\,(\mathbb{R}^2, O)$ be the set of smooth functions $\alpha : \mathbb{R}^2 \to \mathbb{R}$ that are flat at O

Let also $\mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ be the set of all $\mathbb{Z}$-invariant smooth functions $\hat{\alpha} : \mathbb{H} \to \mathbb{R}$ that are flat on $\partial\mathbb{H}$.

Theorem 5.1. *The mapping*

$$P_k : \mathbb{H} \to \mathbb{R}^2, \qquad P_k(\phi, \rho) = (\rho^k \cos\phi, \rho^k \sin\phi)$$

yields a bijection

$$\mathbf{f}_k : \mathrm{Flat}\,(\mathbb{R}^2, O) \to \mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H}), \qquad \mathbf{f}_k(\alpha) = \alpha \circ P_k$$

which is $C^{r,r}_{W,W}$-continuous and its inverse $\mathbf{f}_k^{-1}$ is $C^{(2k+1)r,r}_{W,W}$-continuous for every $r \geq 0$.

Proof. For each $r = 0, \dots, \infty$ let $\mathrm{Func}^{\,r}(\mathbb{R}^2, O)$ be the space of all C^r-functions $\alpha : \mathbb{R}^2 \to \mathbb{R}$ such that $\alpha(O) = 0$, and $\mathrm{Func}^{\,r}(\mathbb{H}, \partial\mathbb{H})$ be the space of $\mathbb{Z}$-invariant C^r-functions $\hat{\alpha} : \mathbb{H} \to \mathbb{R}$ such that $\hat{\alpha}(\partial\mathbb{H}) = 0$.

Then for every $\alpha \in \mathrm{Func}^{\,0}(\mathbb{R}^2, O)$ the function $\hat{\alpha} = \alpha \circ P_k$ is also continuous on $\mathbb{H}$, $\mathbb{Z}$-invariant, and vanish on $\partial\mathbb{H}$, i.e. $\hat{\alpha} \in \mathrm{Func}^{\,0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$. Thus we obtain a well-defined mapping

$$\mathbf{f}_k : \mathrm{Func}^{\,0}(\mathbb{R}^2, O) \to \mathrm{Func}^{\,0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H}), \qquad \mathbf{f}_k(\alpha) = \alpha \circ P_k. \tag{5.1}$$

Conversely, every $\hat{\alpha} \in \mathrm{Func}^{\,0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ yields a unique function $\alpha \in \mathrm{Func}^{\,0}(\mathbb{R}^2, O)$, whence $\mathbf{f}_k$ is a bijection.

Since P_k is smooth, it follows that

$$\mathbf{f}_k\bigl(\mathrm{Func}^{\,\infty}(\mathbb{R}^2, O)\bigr) \ \subset\ \mathrm{Func}^{\,\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$$

and the restriction map

$$\mathbf{f}_k : \mathrm{Func}^{\,\infty}(\mathbb{R}^2, O) \to \mathrm{Func}^{\,\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$$

is $C^{r,r}_{W,W}$-continuous for every $r = 0, \dots, \infty$. But this mapping is not onto, e.g. the second coordinate $\rho : \mathbb{H} \to \mathbb{R}$ being a smooth function is the image of the function $(x^2 + y^2)^{1/2k}$ which is not differentiable at $O \in \mathbb{R}^2$.

Suppose that α is flat at O. Then it is easy to see that $\hat{\alpha}$ is flat at every point of $\partial\mathbb{H}$, i.e. $\mathbf{f}_k(\mathrm{Flat}\,(\mathbb{R}^2, O)) \subset \mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$. The following Lemma 5.2 shows that in fact

$$\mathbf{f}_k(\mathrm{Flat}\,(\mathbb{R}^2, O)) = \mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$$

and the inverse map $\mathbf{f}_k^{-1}$ is $C^{(2k+1)r,r}_{W,W}$-continuous for every $r \geq 0$.

Lemma 5.2. *Suppose that* $\hat{\alpha} \in \mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$*. Let* $\alpha = \mathbf{f}_k^{-1}(\hat{\alpha})$*, and*

$$D\alpha = \frac{\partial^{a+b}\alpha}{\partial x^a \, \partial y^b} \tag{5.2}$$

be a partial derivative of α *of order* $a+b$*.*

(i) *Then* $D\alpha$ *is a sum of finitely many functions of the form*

$$\frac{A \cdot B}{(x^2+y^2)^{s/2k}},$$

where $A : \mathbb{R}^2 \to \mathbb{R}$ *is a smooth function which does not depend on* α *and*

$$B = \mathbf{f}_k^{-1}\left(\frac{\partial^j \hat{\alpha}}{\partial \phi^{j_1} \, \partial \rho^{j_2}} \right), \qquad j = j_1 + j_2 \ \leq \ a+b,$$

and s *is positive integer such that* $s/2k \leq a+b$*. The total number of these functions depends only on* a *and* b *and does not depend on* α*.*

(ii) $D\alpha$ *is a continuous function vanishing at* $O \in \mathbb{R}^2$*. Hence* α *is a smooth function flat at* $O \in \mathbb{R}^2$*, i.e.* $\mathbf{f}_k$ *is a bijection between* $\mathrm{Flat}(\mathbb{R}^2, O)$ *and* $\mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$*.*

(iii) *For every* $r \geq 0$ *and a compact* $K \subset \mathbb{R}^2$ *we have the following estimaiton:*

$$\|\alpha\|_K^r \leq C \|\hat{\alpha}\|_L^{(2k+1)r}, \tag{5.3}$$

where

$$L \ = \ P_k^{-1}(K) \ \ \cap \ \ [0, 2\pi] \times [0, \infty), \tag{5.4}$$

and $C > 0$ *does not depend on* $\hat{\alpha}$*. Whence the inverse mapping* $\mathbf{f}_k^{-1}$ *is* $C^{(2k+1)r,r}_{W,W}$*-continuous.*

Before proving this lemma we establish some formulas.

5.3. **Formulas for** P_k^{-1} **and its derivatives.** Let $(x, y) \in \mathbb{R}^2$. Then $x^2 + y^2 = \rho^{2k}$. For simplicity suppose that $x > 0$, hence

$$\rho = (x^2+y^2)^{\frac{1}{2k}}, \qquad \phi = \arctan(y/x) + 2\pi n,$$

for a certain $n \in \mathbb{Z}$. Therefore

$$\phi'_x = \frac{-y}{x^2+y^2}, \qquad\qquad \phi'_y = \frac{x}{x^2+y^2},$$

$$\rho'_x = \frac{x}{k\,(x^2+y^2)^{1-\frac{1}{2k}}}, \qquad\qquad \rho'_y = \frac{y}{k\,(x^2+y^2)^{1-\frac{1}{2k}}}.$$

Similarly, for every $a, b \geq 0$ there exist smooth functions

$$\mu_i, \nu_i : \mathbb{R}^2 \to \mathbb{R}, \qquad (i = 1, \ldots, a+b),$$

such that

$$\text{(5.5)}\quad \frac{\partial^{a+b}\phi}{\partial x^a \partial y^b} = \sum_{i=1}^{a+b} \frac{\mu_i}{(x^2+y^2)^{a+b}}, \qquad \frac{\partial^{a+b}\rho}{\partial x^a \partial y^b} = \sum_{i=1}^{a+b} \frac{\nu_i}{(x^2+y^2)^{a+b-\frac{1}{2k}}}.$$

These formulas do not depend on a particular choice of the expression of ϕ through x and y.

Proof of Lemma 5.2. (i) First consider the derivative α'_x. Let $z = (x,y) \neq O$. Then in a sufficiently small neighborhood U_z of z we can define an inverse map $P_k^{-1} : U_z \to \mathbb{H}$ such that $\alpha = \hat{\alpha} \circ P_k^{-1}$. Therefore

$$\alpha'_x = (\hat{\alpha}'_\phi \circ P_k^{-1}) \cdot \phi'_x + (\hat{\alpha}'_\rho \circ P_k^{-1}) \cdot \rho'_x.$$

Notice that every partial derivative of $\hat{\alpha} \in \mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ belongs to $\mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ as well, whence by (5.1) this derivative determines a unique continuous function on U_z. Therefore we can write

$$\alpha'_x = \mathbf{f}_k^{-1}(\hat{\alpha}'_\phi) \cdot \phi'_x + \mathbf{f}_k^{-1}(\hat{\alpha}'_\rho) \cdot \rho'_x = \frac{-y \cdot \mathbf{f}_k^{-1}(\hat{\alpha}'_\phi)}{x^2+y^2} + \frac{x \cdot \mathbf{f}_k^{-1}(\hat{\alpha}'_\rho)}{k(x^2+y^2)^{1-\frac{1}{2k}}}.$$

Thus we have obtained a desired presentation. The proof for other partial derivatives of α is similar and we left it to the reader.

(ii) Let us show the continuity of $D\alpha$. Denote

$$D^j \hat{\alpha} = \frac{\partial^j \hat{\alpha}}{\partial \phi^{j_1}\, \partial \rho^{j_2}}.$$

Since $D^j\hat{\alpha}$ is flat on $\partial\mathbb{H}$, it follows that there exists a smooth function $\xi \in \mathrm{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ such that $D^j\hat{\alpha} = \rho^s \xi$. Therefore

$$B = \mathbf{f}_k^{-1}(D^j\hat{\alpha}) = \mathbf{f}_k^{-1}(\rho^s)\ \mathbf{f}_k^{-1}(\xi) = (x^2+y^2)^{s/2k}\ \mathbf{f}_k^{-1}(\xi),$$

whence

$$\text{(5.6)}\qquad \frac{AB}{(x^2+y^2)^{s/2k}} = A\ \mathbf{f}_k^{-1}(\xi)$$

is continuous. Hence $D\alpha$ is continuous as well. Notice that $\xi(\phi, 0) = 0$. Therefore $\mathbf{f}_k^{-1}(\xi_i)(O) = 0$, whence $D\alpha(O) = 0$.

(iii) Let $\alpha = \mathbf{f}_k^{-1}(\hat{\alpha})$. We have to estimate $\|\alpha\|_K^r$. Notice that the subset $L \subset \mathbb{H}$ defined by (5.4) is compact and $P(L) = K$. Therefore

$$\text{(5.7)}\quad \|\mathbf{f}_k^{-1}(\hat{\alpha})\|_K^0 = \|\alpha\|_K^0 = \sup_{x \in K} |\alpha(x)| = \sup_{(\phi,\rho) \in L} |\hat{\alpha}(\phi,\rho)| = \|\hat{\alpha}\|_L^0.$$

By (ii) and (5.6) every partial derivative $D\alpha$ of α of order r can be represented in the form

$$D\alpha = \sum_i A_i \cdot \mathbf{f}_k^{-1}\left(\frac{D^{j_i}\hat{\alpha}}{\rho^{s_i}}\right),$$

where A_i is smooth on all $\mathbb{R}^2$, $D^{j_i}\hat{\alpha}$ is a partial derivative of $\hat{\alpha}$ of order $j_i \le r$, and $s_i \le 2kr$.

Notice that for every i there are constants $C_1, C_2, C_3 > 0$ that do not depend on $\hat{\alpha}$ and such that

$$\left\| \mathbf{f}_k^{-1}\left(\frac{D^{j_i}\hat{\alpha}}{\rho^{s_i}}\right)\right\|_K^0 \overset{(5.7)}{=\!=\!=} \left\|\frac{D^{j_i}\hat{\alpha}}{\rho^{s_i}}\right\|_L^0 \overset{\text{(Lemma 2.3)}}{\le} \tag{5.8}$$

$$\le \; C_1 \left\|D^{j_i}\hat{\alpha}\right\|_L^{s_i} \overset{\text{(Lemma 2.2)}}{\le} C_2 \left\|\hat{\alpha}\right\|_L^{s_i+j_i} \overset{(5.5)}{\le} C_3 \left\|\hat{\alpha}\right\|_L^{(2k+1)r}.$$

Hence there exists $C_4 > 0$ such that

$$\|D\alpha\|_K^0 \; \le \; \sum_i \left\| A_i \cdot \mathbf{f}_k^{-1}\left(\frac{D^{j_i}\hat{\alpha}}{\rho^{k_i}}\right)\right\|_K^0 \; \le \; C_4 \, \|\hat{\alpha}\|_L^{(2k+1)r}.$$

Therefore $\|\alpha\|_K^r \le C\|\hat{\alpha}\|_L^{(2k+1)r}$ for a certain $C > 0$ that depends on K and r but $\hat{\alpha}$. □

Theorem 5.1 is completed.

6. Correspondence between smooth mappings that are ∞-close to the identity

Let $\mathrm{Map}_{\mathbb{Z}}^{\infty}(\mathbb{H}, \partial\mathbb{H})$ be the set of all smooth maps

$$\hat{h} = (\hat{h}_1, \hat{h}_2) : \mathbb{H} \to \mathbb{H},$$

satisfying the following conditions:

(i) $\hat{h}$ is $\mathbb{Z}$-equivariant, i.e.

$$\hat{h}_1(\phi + 2\pi, \rho) = \hat{h}_1(\phi, \rho) + 2\pi, \qquad \hat{h}_2(\phi + 2\pi, \rho) = \hat{h}_2(\phi, \rho). \tag{6.1}$$

(ii) $\hat{h}$ is fixed on $\partial\mathbb{H}$ and $\hat{h}(\mathring{\mathbb{H}}) \subset \mathring{\mathbb{H}}$;

(iii) h is ∞-close to $\mathrm{id}_{\mathbb{H}}$ on $\partial\mathbb{H}$, i.e. the following functions

$$\hat{h}_1(\phi, \rho) - \phi, \qquad \hat{h}_2(\phi, \rho) - \rho$$

are flat on $\partial\mathbb{H}$.

Let also $\mathrm{Map}^{\infty}(\mathbb{R}^2, O)$ be the set of smooth mappings $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h^{-1}(O) = O$ and h is ∞-close to $\mathrm{id}_{\mathbb{R}^2}$ at O.

Lemma 6.1. *Let $\hat{h} = (\hat{h}_1, \hat{h}_2) : \mathbb{H} \to \mathbb{H}$ be a mapping and*

$$\hat{\alpha}(\phi, \rho) = \hat{h}_1(\phi, \rho) - \phi, \qquad \hat{\beta}(\phi, \rho) = \hat{h}_2(\phi, \rho) - \rho.$$

Then $\hat{h}$ is $\mathbb{Z}$-equivariant if and only if the functions $\hat{\alpha}$ and $\hat{\beta}$ are $\mathbb{Z}$-invariant.

Proof. Notice that

$$\begin{aligned}
\hat\alpha(\phi+2\pi,\rho)-\hat\alpha(\phi,\rho) &= \hat h_1(\phi+2\pi,\rho)-\phi-2\pi-(\hat h_1(\phi,\rho)-\phi)\\
&= \hat h_1(\phi+2\pi,\rho)-\hat h_1(\phi,\rho)-2\pi,\\
\hat\beta(\phi+2\pi,\rho)-\hat\beta(\phi,\rho) &= \hat h_2(\phi+2\pi,\rho)-\rho-(\hat h_2(\phi,\rho)-\rho)\\
&= \hat h_2(\phi+2\pi,\rho)-\hat h_2(\phi,\rho).
\end{aligned}$$

These identities together with (6.1) imply our statement. □

Theorem 6.2. *The mapping P_k yields a $C^{r,r}_{W,W}$-continuous bijection*

$$\mathbf{m}_k : \mathrm{Map}^{\infty}(\mathbb{R}^2, O) \to \mathrm{Map}^{\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$$

such that its inverse $\mathbf{m}_k^{-1}$ is $C^{(2k+1)r,r}_{W,W}$-continuous for every $r \geq 0$.

Proof. Let $\mathrm{Map}^{0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ be the set of all continuous, $\mathbb{Z}$-equivariant mappings $\hat h : \mathbb{H} \to \mathbb{H}$ that are fixed on $\partial\mathbb{H}$ and $\hat h(\mathring{\mathbb{H}}) \subset \mathring{\mathbb{H}}$.

Let also $\mathrm{Map}^{0}(\mathbb{R}^2, O)$ be the set of all continuous maps $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h^{-1}(O) = O$.

Then every $\hat h \in \mathrm{Map}^{0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ yields a unique $h \in \mathrm{Map}^{0}(\mathbb{R}^2, O)$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\hat h} & \mathbb{H} \\
{\scriptstyle P_k}\downarrow & & \downarrow{\scriptstyle P_k} \\
\mathbb{R}^2 & \xrightarrow{h} & \mathbb{R}^2
\end{array}$$

i.e. $h \circ P_k = P_k \circ \hat h$. In the coordinate form this means that

$$\begin{array}{lrcl}
(6.2) & h_1(\rho^k\cos\phi, \rho^k\sin\phi) & = & \hat h_2(\phi,\rho)^k \cdot \cos\hat h_1(\phi,\rho) \\
 & h_2(\rho^k\cos\phi, \rho^k\sin\phi) & = & \hat h_2(\phi,\rho)^k \cdot \sin\hat h_1(\phi,\rho).
\end{array}$$

For such a pair h and $\hat h$ we will use the following notations:

$$\hat\alpha(\phi,\rho) = \hat h_1(\phi,\rho) - \phi, \qquad \hat\beta(\phi,\rho) = \hat h_2(\phi,\rho) - \rho, \tag{6.3}$$

$$\gamma(x,y) = h_1(x,y) - x, \qquad \delta(x,y) = h_2(x,y) - y. \tag{6.4}$$

Thus the correspondence $\hat h \mapsto h$ is a well-defined mapping

$$\mathbf{m}'_k : \mathrm{Map}^{0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H}) \to \mathrm{Map}^{0}(\mathbb{R}^2, O).$$

Our aim is to prove that $\mathbf{m}'_k$ yields a bijection

$$\mathbf{m}_k^{-1} : \mathrm{Map}^{\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H}) \to \mathrm{Map}^{\infty}(\mathbb{R}^2, O).$$

First let us show that the image of $\mathbf{m}'_k$ includes $\mathrm{Map}^{\infty}(\mathbb{R}^2, O)$. Indeed, let $h \in \mathrm{Map}^{\infty}(\mathbb{R}^2, O)$. Since h is C^1 (actually C^{∞}) and 1-close to the identity at O (actually ∞-close), we have that the tangent map

$$T_O h : T_O\mathbb{R}^2 \to T_O\mathbb{R}^2$$

is the identity. Therefore h induces a unique mapping $\hat{h} : \mathbb{H} \to \mathbb{H}$ fixed on $\partial\mathbb{H}$. Moreover, since $h^{-1}(O) = O$, we obtain that $\hat{h}(\mathring{\mathbb{H}}) = \mathring{\mathbb{H}}$, whence $\hat{h} \in \mathrm{Map}^{0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$, and thus $\mathbf{m}'_k(\hat{h}) = h$.

Also notice that a uniqueness of such $\hat{h}$ implies that we have a well-defined map

$$\mathbf{m}_k : \mathrm{Map}^{\infty}(\mathbb{R}^2, O) \to \mathrm{Map}^{0}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$$

inverse to $\mathbf{m}'_k$.

It remains to prove the following lemma:

Lemma 6.3. $\mathbf{m}_k(\mathrm{Map}^{\infty}(\mathbb{R}^2, O)) = \mathrm{Map}^{\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$. *Moreover, for every $r \geq 0$ the restriction map*

$$\mathbf{m}_k : \mathrm{Map}^{\infty}(\mathbb{R}^2, O) \to \mathrm{Map}^{\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$$

is $C^{r,r}_{W,W}$-continuous, while its inverse

$$\mathbf{m}_k^{-1} : \mathrm{Map}^{\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H}) \to \mathrm{Map}^{\infty}(\mathbb{R}^2, O)$$

is $C^{(2k+1)r,r}_{W,W}$-continuous.

Proof. Let $h \in \mathrm{Map}^{\infty}(\mathbb{R}^2, O)$ and $\hat{h} = \mathbf{m}_k(h)$. It suffices to prove that $\hat{h}$ is smooth and ∞-close to $\mathrm{id}_{\mathbb{H}}$ on $\partial\mathbb{H}$ in a neighborhood of $(0, 0) \in \mathbb{H}$.

Since $h(O) = O$ and h is ∞-close to $\mathrm{id}_{\mathbb{R}^2}$ at O we have that

$$h_1(x, y) = x + xa_1 + yb_1, \qquad h_2(x, y) = y + xa_2 + yb_2, \tag{6.5}$$

where $a_1, a_2, b_1, b_2 \in \mathrm{Flat}\,(\mathbb{R}^2, O)$.

Then it follows from (6.2) and (6.5) that

$$(h_1 \circ P_k)^2 + (h_2 \circ P_k)^2 = \hat{h}_2^2 = \rho^{2k} \cdot (1 + \omega(\phi, \rho)),$$

$$2 \cdot (h_1 \circ P_k) \cdot (h_2 \circ P_k) = \hat{h}_2^{2k} \cdot \sin 2\hat{h}_1 = \rho^{2k} \cdot (\sin 2\phi + \xi(\phi, \rho))$$

where $\omega, \xi : \mathbb{H} \to \mathbb{R}$ are smooth functions flat on $\partial\mathbb{H}$. Hence

$$\sin 2\hat{h}_1 = \frac{\sin 2\phi + \xi}{1 + \omega} = (\sin 2\phi + \xi)(1 - \omega + \omega^2 - \cdots) = \sin 2\phi + \psi,$$

where ψ is smooth in a neighborhood of $(0, 0) \in \mathbb{H}$ and flat on $\partial\mathbb{H}$. Therefore by (2.2)

$$\hat{h}_1 = \frac{1}{2}\,\arcsin(\sin 2\phi + \psi) \xlongequal{(2.2)} \phi + \psi \cdot \tau(\phi, \rho),$$

where τ is smooth in a neighborhood of $(0,0) \in \mathbb{H}$. Hence $\hat{h}_1(\phi,\rho) - \phi$ is smooth in a neighborhood of $(0,0) \in \mathbb{H}$ and flat on $\partial\mathbb{H}$.

It remains to prove a smoothness of $\hat{h}_2$ at every point $(\phi_0, 0)$. Let $A = \cos\phi_0$, $B = \sin\phi_0$. Then it follows from (6.2) and (6.5) that

$$A \cdot h_1 \circ P_k + B \cdot h_1 \circ P_k \overset{(6.2)}{=\!=} \hat{h}_2^k \cdot (A\cos\hat{h}_1 + B\sin\hat{h}_1) = \\ = \hat{h}_2^k \cos(\hat{h}_1 - \phi_0).$$

$$A \cdot h_1 \circ P_k + B \cdot h_1 \circ P_k \overset{(6.5)}{=\!=} \rho^k(A\cos\phi + B\sin\phi + c) = \\ = \rho^k(\cos(\phi - \phi_0) + c),$$

where $c \in \operatorname{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$. Hence

$$\hat{h}_2(\phi,\rho) = \rho \cdot \underbrace{\sqrt[k]{\frac{\cos(\phi-\phi_0)+c}{\cos(\hat{h}_1-\phi_0)}}}_{\eta} = \rho\cdot\eta(\phi,\rho) \tag{6.6}$$

Since $\hat{h}_1$ is smooth and $\hat{h}_1 - \phi$ is flat on $\partial\mathbb{H}$, it follows that in a neighborhood of $(\phi_0, 0)$ the function η is smooth and $\eta - 1$ is flat. Hence

$$\hat{h}_2 = \rho + \hat{\beta},$$

where $\beta \in \operatorname{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$. It also follows that $\mathbf{m}_k$ is $C^{r,r}_{W,W}$-continuous.

Consider now the map $\mathbf{m}_k^{-1}$. Let $\hat{h} = (\hat{h}_1, \hat{h}_2) \in \operatorname{Map}^{\infty}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ and

$$h = \mathbf{m}_k^{-1}(\hat{h}) = (h_1, h_2) \ \in \ \operatorname{Map}^0(\mathbb{R}^2, O).$$

By assumption $\hat{\alpha}$ and $\hat{\beta}$ are flat on $\partial\mathbb{H}$ and by Lemma 6.1 they are $\mathbb{Z}$-invariant, whence $\hat{\alpha}, \hat{\beta} \in \operatorname{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$. We have to show that γ and δ are smooth and flat at $O \in \mathbb{R}^2$. Due to Theorem 5.1 it suffices to establish that $\gamma \circ P_k$ and $\delta \circ P_k$ belong to $\operatorname{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$.

By (2.1) there are smooth functions $\mu, \nu : \mathbb{H} \to \mathbb{R}$ such that

$$\cos\hat{h}_1 = \cos(\phi + \hat{\alpha}) = \cos\phi + \hat{\alpha}\cdot\mu(\phi,\hat{\alpha}),$$
$$\sin\hat{h}_1 = \sin(\phi + \hat{\alpha}) = \sin\phi + \hat{\alpha}\cdot\nu(\phi,\hat{\alpha}).$$

Evidently, μ and ν are $\mathbb{Z}$-invariant. Also notice that

$$\hat{h}_2^k = (\rho + \hat{\beta})^k = \rho^k + \hat{\beta}_1,$$

for some $\hat{\beta}_1 \in \operatorname{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$. Hence

$$\begin{aligned} \gamma \circ P_k(\phi,\rho) &= (\rho^k + \hat{\beta}_1)(\cos\phi + \hat{\alpha}\cdot\mu(\phi,\hat{\alpha})) - \rho^k\cos\phi = \\ &= \hat{\beta}_1 \cdot \cos\phi + (\rho^k + \hat{\beta}_1)\cdot\hat{\alpha}\cdot\mu(\phi,\hat{\alpha}), \\ \delta \circ P_k(\phi,\rho) &= \hat{\beta}_1 \cdot \sin\phi + (\rho^k + \hat{\beta}_1)\cdot\hat{\alpha}\cdot\nu(\phi,\hat{\alpha}). \end{aligned} \tag{6.7}$$

Since $\hat{\alpha}, \hat{\beta} \in \operatorname{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$, we see that $\gamma \circ P_k, \delta \circ P_k \in \operatorname{Flat}_{\mathbb{Z}}(\mathbb{H}, \partial\mathbb{H})$ as well.

It remains to note that the mapping $\mathbf{m}_k^{-1}$ coincides with the following sequence of correspondences:

$$\hat{h} \xmapsto{(6.3)} (\hat{\alpha}, \hat{\beta}) \xmapsto{(6.7)} (\gamma \circ P, \delta \circ P) \xmapsto{\mathbf{f}_k} (\gamma, \delta) \xmapsto{(6.3)} h,$$

in which for every $r \geq 0$ the first and second arrows are $C^{r,r}_{W,W}$-continuous and by Theorem 5.1 the third one is $C^{(2k+1)r,r}_{W,W}$-continuous. Hence $\mathbf{m}_k^{-1}$ is $C^{(2k+1)r,r}_{W,W}$-continuous for every $r \geq 0$. $\square$

Theorem 6.2 is completed.

7. Proof of Proposition 3.4.

Let G be a smooth vector field, defined in a neighborhood V of the origin $O \in \mathbb{R}^2$. Suppose that G has property $(*)$ at O. Therefore we can assume that $G = \eta H$, where $\eta : \mathbb{R}^2 \to \mathbb{R} \setminus \{0\}$ is everywhere nonzero smooth function and $H = (-g'_y, g'_x)$ is a Hamiltonian vector field of a certain homogeneous polynomial $g : \mathbb{R}^2 \to \mathbb{R}$ of degree $p + 1 \geq 2$ having no multiple factors.

Denote by $\mathbf{G}$ the corresponding local flow of G.

For every $h \in \mathcal{E}_\infty(G, V, O)$ we have to find a smooth function

$$\alpha : V \to \mathbb{R}$$

which is flat at O and such that

$$h(z) = \mathbf{G}(z, \alpha(z)).$$

Let $P : \mathbb{H} \to \mathbb{R}^2$ be the map defining polar coordinates, i.e.

$$P(\phi, \rho) = (\rho \cos \phi, \rho \sin \phi).$$

Thus $P = P_1$ in the notation of Section 4.

Set $U = P^{-1}(V)$.

Let $\operatorname{Flat}(V, O)$ be the space of smooth functions $V \to \mathbb{R}$ which are flat at O, and $\operatorname{Flat}_{\mathbb{Z}}(U, \partial\mathbb{H})$ be the space of smooth $\mathbb{Z}$-invariant functions $U \to \mathbb{R}$ which are flat on $\partial\mathbb{H}$.

Denote by $\operatorname{Map}(V, \mathbb{R}^2, O)$ the space of smooth maps $h : V \to \mathbb{R}^2$ such that $h^{-1}(O) = O$ and h is ∞-close to id_V at O. Finally, let $\operatorname{Map}_{\mathbb{Z}}(U, \mathbb{H}, \partial\mathbb{H})$ be the space of smooth $\mathbb{Z}$-equivariant mappings $\hat{h} : U \to \mathbb{H}$ such that $\hat{h}^{-1}(\partial\mathbb{H}) = \partial\mathbb{H}$ and $\hat{h}$ is ∞-close to id_U at every points of $\partial\mathbb{H}$.

Then it follows from Theorems 5.1 and 6.2 that the mapping P yields the following bijections $\mathbf{f}_1$ and $\mathbf{m}_1$ which for simplicity we denote by $\mathbf{f}$ and $\mathbf{m}$ respectively:

$$\mathbf{f} : \mathrm{Flat}\,(V, O) \to \mathrm{Flat}_{\mathbb{Z}}(U, \partial\mathbb{H}),$$
$$\mathbf{m} : \mathrm{Map}\,(V, \mathbb{R}^2, O) \to \mathrm{Map}_{\mathbb{Z}}(U, \mathbb{H}, \partial\mathbb{H}).$$

Let F be the lifting of the vector field G from V to U via P. Denote by $\mathcal{E}_\infty(F, U, \partial\mathbb{H})$ the subset of $\mathcal{E}(F, U)$ consisting of mappings that are ∞-close to $\mathrm{id}_{\mathbb{H}}$ on $\partial\mathbb{H}$. Moreover, let $\mathcal{E}_\infty(F, U, \partial\mathbb{H})_{\mathbb{Z}}$ be the subset of $\mathcal{E}_\infty(F, U, \partial\mathbb{H})$ consisting of $\mathbb{Z}$-equivariant maps. Then we have the following inclusions:

$$\begin{array}{ccc} \mathrm{Map}\,(V, \mathbb{R}^2, O) & \supset & \mathcal{E}_\infty(G, V, O) \\ {\scriptstyle \mathbf{m}}\big\downarrow & & \\ \mathrm{Map}_{\mathbb{Z}}(U, \mathbb{H}, \partial\mathbb{H}) & \supset & \mathcal{E}_\infty(F, U, \partial\mathbb{H})_{\mathbb{Z}}. \end{array}$$

Lemma 7.1. $\mathbf{m}\big(\,\mathcal{E}_\infty(G, V, O)\,\big) = \mathcal{E}_\infty(F, U, \partial\mathbb{H})_{\mathbb{Z}}$.

Proof. Let

$$h \in \mathcal{E}_\infty(G, V, O) \qquad\qquad \hat{h} = \mathbf{m}(h) \in \mathrm{Map}_{\mathbb{Z}}(U, \mathbb{H}, \partial\mathbb{H}).$$

We have to show that $\hat{h} \in \mathcal{E}_\infty(F, U, \partial\mathbb{H})_{\mathbb{Z}}$, i.e.

(i) $\hat{h}$ is a diffeomorphism in a neighborhood of every singular point point $z \in \Sigma_F = \partial\mathbb{H}$ of F;
(ii) $\hat{h}(\hat{\omega} \cap U) \subset \hat{\omega}$ for every orbit $\hat{\omega}$ of F.

Proof of (i). Since h is ∞-close to $\mathrm{id}_{\mathbb{R}^2}$ at O, it follows from Theorem 6.2 that $\hat{h}$ is ∞-close to the identity on $\Sigma_F = \partial\mathbb{H}$. Therefore for every point $z \in \partial\mathbb{H}$ the corresponding tangent map $T_z\hat{h} : T_z\mathbb{H} \to T_z\mathbb{H}$ is identity and thus it is nondegenerate.

Proof of (ii). Let $\hat{\omega}$ be an orbit of F and $\omega = P(\hat{\omega})$ be the corresponding orbit of G. Then by definition $h(\omega \cap V) \subset \omega$. Hence $\hat{h}(\hat{\omega} \cap U)$ is included in some orbit $\hat{\omega}_1$ of F which is also mapped onto ω by P, i.e. $P(\hat{\omega}_1) = \omega$.

We have to show that $\hat{\omega} = \hat{\omega}_1$. Actually this follow from the structure of orbits of G.

Indeed, suppose that g is a product of definite quadratic forms, i.e. $g(z) \neq 0$ for $z \neq 0$. Then the structure of the orbits of F and G for this case is shown in Figure 4.2. It follows from this figure that $\hat{\omega} = P^{-1}(\omega)$, whence $\hat{\omega} = \hat{\omega}_1$.

Suppose that g has linear factors. Then, see Figure 4.1, the set $g^{-1}(0)$ is a union of $2l$ rays $T_0, \ldots, T_{2l-1}$ for $i = 1, \ldots, l$ starting at the origin O and such that T_i and $T_{i+l \bmod 2l}$ belong to the same straight

line. Moreover, the set $P^{-1}\circ g^{-1}(O)$ is a union of $\partial\mathbb{H}$ together with countable set of vertical half-lines $\hat{T}_j$, $(j\in\mathbb{Z})$. We can assume that $P(\hat{T}_j)=T_{j \bmod 2l}$.

Since $h(T_i)=T_i$, it follows that $\hat{h}(\hat{T}_j)$ for all i and j. Therefore P yields a bijection between the orbits of G laying in the angles between T_i and T_{i+1} and orbits of F laying between $\hat{T}_{i+2ls}$ and $\hat{T}_{i+1+2ls}$, $(s\in\mathbb{Z})$. Hence $\hat{\omega}=\hat{\omega}_1$.

Thus $\mathbf{m}\bigl(\mathcal{E}_\infty(G,V,O)\bigr)\subset\mathcal{E}_\infty(F,U,\partial\mathbb{H})_{\mathbb{Z}}$.

Conversely, let $\hat{h}\in\mathcal{E}_\infty(F,U,\partial\mathbb{H})_{\mathbb{Z}}$ and $h=\mathbf{m}^{-1}(\hat{h})\in\mathrm{Map}\,(V,\mathbb{R}^2,O)$. We have to show that $h\in\mathcal{E}_\infty(G,V,O)$. Since h is ∞-close to $\mathrm{id}_{\mathbb{R}^2}$ at O, we obtain that h is a local diffeomorphism at every (actually unique) singular point of G. Moreover, let ω be any orbit of G and $\hat{\omega}$ be an orbit of F such that $\omega=P(\hat{\omega})$. Then by definition $\hat{h}(\hat{\omega}\cap U)\subset\hat{\omega}$.

Since $P\circ\hat{h}=h\circ P$, we obtain that

$$h(\omega\cap V)\ \subset\ h\circ P(\hat{\omega}\cap U)\ =\ P\circ\hat{h}(\hat{\omega}\cap U)\ \subset\ P(\hat{\omega})\ =\ \omega.$$

Thus $\mathcal{E}_\infty(F,U,\partial\mathbb{H})_{\mathbb{Z}}\subset\mathbf{m}\bigl(\mathcal{E}_\infty(G,V,O)\bigr)$. $\square$

It remains to prove the following statement:

Proposition 7.2. *Suppose that g has property $(*)$. Then there exists a unique mapping*

$$\psi:\mathcal{E}_\infty(F,U,\partial\mathbb{H})_{\mathbb{Z}}\to\mathrm{Flat}_{\mathbb{Z}}(U,\partial\mathbb{H})$$

such that

$$\hat{h}(x)=\mathbf{F}(x,\psi(\hat{h})(x))$$

for all $\hat{h}\in\mathcal{E}_\infty(F,U,\partial\mathbb{H})_{\mathbb{Z}}$. This map is $C^{r+p,r}_{W,W}$-continuous.

Corollary 7.3. *Define the mapping $\Psi:\mathcal{E}_\infty(G,V,O)\to\mathrm{Flat}\,(V,O)$ by $\Psi=\mathbf{f}^{-1}\circ\psi\circ\mathbf{m}$, i.e. so that the following diagram becomes commutative:*

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathbb{Z}}(U,\mathbb{H},\partial\mathbb{H}) & \supset & \mathcal{E}_\infty(F,U,\partial\mathbb{H})_{\mathbb{Z}} & \xrightarrow{\psi} & \mathrm{Flat}_{\mathbb{Z}}(U,\partial\mathbb{H}) \\
{\scriptstyle\mathbf{m}}\big\uparrow & & {\scriptstyle\mathbf{m}}\big\uparrow & & \big\uparrow{\scriptstyle\mathbf{f}} \\
\mathrm{Map}\,(V,\mathbb{R}^2,O) & \supset & \mathcal{E}_\infty(G,V,O) & \xrightarrow{\Psi} & \mathrm{Flat}\,(V,O)
\end{array}$$

Then Ψ satisfies the statement of Proposition 3.4.

Proof of Corollary. Indeed, let $h\in\mathcal{E}_\infty(G,V,O)$,

$$\hat{h}=\mathbf{m}(h)\in\mathcal{E}_\infty(F,U,\partial\mathbb{H})_{\mathbb{Z}},\qquad \hat{\alpha}=\psi(\hat{h})\in\mathrm{Flat}_{\mathbb{Z}}(U,\partial\mathbb{H}).$$

So

$$\hat{h}(a)=\mathbf{F}(a,\hat{\alpha}(a)),\qquad\forall a\in U.$$

Set

$$\alpha \;=\; \mathbf{f}^{-1}(\hat{\alpha}) \;=\; \mathbf{f}^{-1}\circ\psi\circ\mathbf{m}(h) \;\in\; \mathrm{Flat}\,(V,O),$$

thus $\hat{\alpha}=\alpha\circ P$. First we have to show that

$$h(b)=\mathbf{G}(b,\alpha(b)), \qquad \forall b\in V.$$

Let $a\in U$ and $b\in V$ be such that $b=P(a)$. Then

$$\begin{aligned} h(b) = h\circ P(a) = P\circ\hat{h}(a) = P\circ\mathbf{F}(a,\hat{\alpha}(a)) = \\ = \mathbf{G}(P(a),\hat{\alpha}(a)) = \mathbf{G}(P(a),\alpha\circ P(a)) = \mathbf{G}(b,\alpha(b)). \end{aligned}$$

It remains to prove continuity of Ψ.

Notice that for every $r\geq p$ the mapping $\mathbf{m}$ is $C^{r,r}_{W,W}$-continuous, ψ is $C^{r,r-p}_{W,W}$-continuous, and $\mathbf{f}^{-1}$ is $C^{r-p,[(r-p)/3]}_{W,W}$-continuous, where $[t]$ is the integer part of $t\in\mathbb{R}$. Hence Ψ is $C^{r,[(r-p)/3]}_{W,W}$-continuous of all $r\geq p$.

Replacing r by $3r+p$ we obtain that Ψ is $C^{3r+p,r}_{W,W}$-continuous. $\square$

Thus Proposition 3.4 and therefore Theorem 3.2 are proved modulo Proposition 7.2.

Remark 7.4. Let $A\in\mathrm{Flat}\,(U,\partial\mathbb{H})$, i.e. A is flat on $\partial\mathbb{H}$. Then it follows from the Hadamard lemma that for every $t\in\mathbb{N}$ there exists $A_t\in\mathrm{Flat}\,(U,\partial\mathbb{H})$ such that $A=\rho^t A_t$.

Proof of Proposition 7.2. Let $\hat{h}=(\hat{h}_1,\hat{h}_2)\in\mathcal{E}_\infty(F,U,\partial\mathbb{H})$. Since all orbits of F in $\overset{\circ}{\mathbb{H}}$ are non-closed, it follows that for every $z\in\overset{\circ}{\mathbb{H}}$ there exists a unique number $\psi(z)\in\mathbb{R}$ such that

$$\hat{h}(z)=\mathbf{G}(z,\psi(z)).$$

Thus we get a shift-function $\psi:\overset{\circ}{\mathbb{H}}\to\mathbb{R}$ for $\hat{h}$. Moreover, it follows from (1.4) that this function is smooth on $\overset{\circ}{\mathbb{H}}$.

Define ψ on $\partial\mathbb{H}$ by $\psi(z)=0$ for $z\in\partial\mathbb{H}$. We have show that this extension is smooth of $\mathbb{H}$ and flat on $\partial\mathbb{H}$.

Let $\phi_0\in\partial\mathbb{H}$. Then by Lemma 4.1

$$g\circ P(\phi,\rho)=\rho^{p+1}(\phi-\phi_0)^a\gamma(\phi),$$

for some $a\geq 0$ depending on ϕ_0 and a smooth function $\gamma:\mathbb{R}\to\mathbb{R}$ such that $\gamma(\phi_0)\neq 0$.

Moreover, since g has property $(*)$, it follows from Corollary 4.5 that a is either 0 or 1.

Consider two cases. Not loosing generality, we can also assume that $\phi_0=0$.

1) Suppose that $a = 0$, i.e.

$$g \circ P(\phi, \rho) = \rho^{p+1}\gamma(\phi),$$

is a neighborhood of $(0,0) \in \mathbb{H}$. Equivalently, this means that g is not divided by y. Then by (4.4) of Lemma 4.4 we have that

$$F_1(\phi, \rho) = \rho^{p-1}\,\gamma_1(\phi).$$

Since $\hat{h}_1 - \phi$ and $\hat{h}_2 - \rho$ are flat on $\partial\mathbb{H}$, they are divided by ρ, whence we can write

$$\hat{h}_1(\phi, \rho) = \phi + A(\phi, \rho), \qquad \hat{h}_2(\phi, \rho) = \rho + \rho B(\phi, \rho),$$

where $A, B \in \mathrm{Flat}\,(U, \partial\mathbb{H})$.

Notice that F defines a the following system of ODE:

$$\begin{cases} \dot{\phi} = F_1(\phi, \rho) \\ \dot{\rho} = F_2(\phi, \rho). \end{cases}$$

Whence $dt = \frac{d\phi}{F}$. Therefore the time $\psi(\phi, \rho)$ between the points (ϕ, ρ) and $\hat{h}(\phi, \rho)$ can be calculated by the following formula:

$$\psi(\phi, \rho) = \int\limits_{\phi}^{\hat{h}_1(\phi,\rho)} \frac{d\theta}{\rho^{p-1}\,\gamma(\theta)}.$$

We will show that ψ is smooth in a neighborhood of $(0,0) \in \mathbb{H}$. It suffices to prove that ψ has smooth partial derivatives of the first order which are flat on $\partial\mathbb{H}$.

An easy calculation shows that

$$\psi'_{\phi}(\phi, \rho) = \frac{(\hat{h}_1)'_{\phi}}{\hat{h}_2^{p-1} \cdot \gamma(\hat{h}_1)} - \frac{1}{\rho^{p-1} \cdot \gamma}, \qquad \psi'_{\rho}(\phi, \rho) = \frac{(\hat{h}_1)'_{\rho}}{\hat{h}_2^{p-1}\gamma(\hat{h}_1)}.$$

Notice that

$$(\hat{h}_1)'_{\phi} = 1 + A'_{\phi}, \qquad (\hat{h}_1)'_{\rho} = A'_{\rho}.$$

Moreover,

$$\hat{h}_2^{p-1} = \rho^{p-1}(1 + \bar{B}), \qquad \gamma(\hat{h}_1(\phi, \rho)) = \gamma(\phi)(1 + C), \tag{7.1}$$

for some $\bar{B}, C \in \mathrm{Flat}\,(U, \partial\mathbb{H})$. Hence

$$\begin{aligned} \psi'_{\phi}(\phi, \rho) &= \frac{1 + A'_{\phi}}{\rho^{p-1}(1 + \bar{B})\gamma(\hat{h}_1)} - \frac{1 + C}{\rho^{p-1}\gamma(\hat{h}_1)} = \\ &= \frac{\overbrace{A'_{\phi} - \bar{B} - C - \bar{B}C}^{D}}{\rho^{p-1}(1 + \bar{B})\gamma(\hat{h}_1)} = \frac{D/\rho^{p-1}}{(1 + \bar{B})\gamma(\hat{h}_1)}. \end{aligned} \tag{7.2}$$

Since $D \in \operatorname{Flat}(U, \partial\mathbb{H})$, it follows from the Hadamard lemma, see Remark 7.4, that D/ρ^{p-1} and therefore $\psi'_{\phi}(\phi,\rho)$ belong to $\operatorname{Flat}(U, \partial\mathbb{H})$.

Similarly,

$$\psi'_{\rho}(\phi,\rho) = \frac{A'_{\rho}}{\rho^{p-1}(1+\bar{B})\gamma(\hat{h}_1)} = \frac{A'_{\rho}/\rho^{p-1}}{(1+\bar{B})\gamma(\hat{h}_1)}. \tag{7.3}$$

Again this function is smooth since $A'_{\rho} \in \operatorname{Flat}(U, \partial\mathbb{H})$.

2) Suppose that $a = 1$. Then $g = yR$, where $R(x,0) \neq 0$ and by (4.5) of Lemma 4.4

$$F_2(\phi,\rho) = \rho^p\,\gamma_2(\phi).$$

Since $F_1(0,\rho) = 0$, we see that the half-axis $\{\phi = 0, \rho > 0\}$ is the orbit of F. Therefore $\hat{h}$ preserves this half-axis, i.e. $\hat{h}_1(0,\rho) = 0$, whence by the Hadamard lemma we obtain that

$$\hat{h}_1(\phi,\rho) = \phi + \phi A(\phi,\rho), \qquad \hat{h}_2(\phi,\rho) = \rho + \rho B(\phi,\rho)$$

for certain $A, B \in \operatorname{Flat}(U, \partial\mathbb{H})$. Therefore

$$\psi(\phi,\rho) = \int\limits_{\rho}^{\hat{h}_2(\phi,\rho)} \frac{d\rho}{\rho^p\,\gamma(\phi)}.$$

Then similarly to the previous case it can be shown that

$$\psi'_{\phi}(\phi,\rho) = \frac{B'_{\phi}/\rho^p}{(1+\bar{B})\gamma(\hat{h}_1)}, \tag{7.4}$$

and

$$\psi'_{\rho}(\phi,\rho) = \frac{E/\rho^p}{(1+\bar{B})\gamma(\hat{h}_1)}, \tag{7.5}$$

where similarly to (7.1) $\bar{B}, C, E$ are defined by

$$\hat{h}_2^p = \rho^p(1+\bar{B}), \qquad \gamma(\hat{h}_1(\phi,\rho)) = \gamma(\phi)(1+C),$$

$$E = B'_{\rho} - \bar{B} - C - \bar{B}C$$

and belong to $\operatorname{Flat}(U, \partial\mathbb{H})$. Hence $\psi \in \operatorname{Flat}(U, \partial\mathbb{H})$ as well.

It remains to prove continuity of the correspondence $\hat{h} \mapsto \psi$. Notice that the expressions for ψ'_{ϕ} and ψ'_{ρ} include division by ρ^p and the operators $\partial/\partial\phi$, and $\partial/\partial\rho$. Recall that by Lemmas 2.2 and 2.3 the division by ρ and differentiating by ϕ and ρ are $C^{r+1,r}_{W,W}$-continuous.

It follows from formulas (7.2), (7.3), (7.4), and (7.5) that there exists $d > 0$ and a closed ball $K \subset V$ containing $O \in \mathbb{R}^2$ such that the

absolute values of denominators of these expressions are greater than $2d$ at every point of K. Put

$$L = P^{-1}(K) \cap [0, 2\pi] \times [0, \infty).$$

Then it follows from expressions for ψ'_ϕ and ψ'_ρ and Lemmas 2.2 and 2.3 that for every $r \geq 0$ and $\varepsilon > 0$ there exists $\delta \in (0, d)$ such that the inequality

$$\|\hat{h} - q\|_K^{r+p+1} < \delta \qquad \text{implies} \qquad \|\psi(\hat{h}) - \psi(q)\|_L^{r+1} < \varepsilon.$$

Hence the correspondence $\hat{h} \mapsto \psi$ is $C^{r+p,r}_{W,W}$-continuous for all $r \geq 0$. We leave the details to the reader.

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